

# Quantum-Classical Transitions in Complex Networks

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## Abstract

The inherent properties of specific physical systems can be used as a metaphor for investigating the behavior of complex networks. This insight has already been put into practice by previous works, e.g., studying the network evolution in terms of phase transitions of quantum gases or representing distances among nodes as they were particle energies. In this work, we show that the emergence of a scale-free structure in complex networks can be represented in terms of a quantum-classical transition for quantum gases. In particular, we propose a model of fermionic networks that allows to investigate the network evolution and its dependence with the system temperature. Simulations, performed in accordance with the cited model, clearly highlight the separation between simple random vs. scale-free networks, with full correspondence with classical vs. quantum regions for quantum gases.

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## I. INTRODUCTION

Many natural and man-made complex systems can be modeled by complex networks, i.e., biological cells and their interactions, social dynamics, world wide web, and every other system containing a notable number of interacting elements (Ref. [1][2][3]). In general, complex networks allow to represent systems at a non-equilibrium state, meaning that new nodes and/or new links can be added over time during an evolutionary, often irreversible, process. In Ref. [1], Barabasi illustrated the central role of statistical mechanics for analysing these networks. One of the first models for random graphs dynamics has been developed by Erdos and Renyi, see Ref. [4]. This kind of networks, called *ER graphs*, have low clustering coefficient and a binomial degree distribution of nodes, converging to a Poissonian distribution for large number of nodes. Later on, in their *WS model*, Watts and Strogatz Ref. [5] interpolated *ER graphs* with a regular ring lattice, achieving networks characterized by small average shortest path length, high clustering coefficient, and homogeneous degree distribution of nodes. Observing that in many real networks the degree distribution of nodes follows a power-law, Barabasi developed the *BA model* Ref. [1], defining the concept of *scale-free* networks. The power-law equation is the following:

$$P(k) \sim k^{-\gamma} \tag{1}$$

where  $k$  is the node degree and  $\gamma$  is a parameter usually in the range  $(2, 3)$ . A notable consequence is that, in these networks, only few nodes (called hubs) have a big number of links. The evolutionary process of networks has been studied by many authors, often resorting to theoretical physics. For the sake of brevity, let us cite only few. Bianconi and Barabasi Ref. [6] compared Bose-Einstein Condensation phenomena (BEC) to winner-takes-all policies; Bianconi Ref. [7] discussed the symmetric construction of bosonic and fermionic networks, asserting that the former networks are scale-free and the latter are growing Calay trees. Kriukov et al. Ref. [8] developed a geometric framework to study the structure and functions of complex networks, interpreting edges as non-interacting fermions whose energies are hyperbolic distances between nodes. Shen Yi et al. Ref. [9] discussed an inverse approach to network evolution defining a relation with their model and Fermi-Dirac statistics. Baronchelli et al. Ref. [10] defined a framework using bosonic reaction-diffusion processes, with the aim of analysing dynamical systems on complex networks. Perseguers et

al. Ref. [11] developed a model of quantum complex networks, drawing a link between the field of complex networks and that of quantum computing.

In this work we propose a theoretical model of network evolution based on Fermi-Dirac statistics, obtained by mapping complex networks to fermionic gases. The proposed model, which can be seen as dual with respect to bosonic networks (Ref. [6]), shows that the emergence of a scale-free structure can be represented in terms of a quantum-classical transition for quantum gases. In particular, at low temperatures networks are scale-free, whereas at high temperatures they become simple random networks. As a result of several simulations, we computed a separation line between scale-free and simple random regions, in a temperature-density diagram. Notably, this line performs a separation that mimics the one observable between classical and quantum regions of quantum gases. The remainder of the paper is organized as follows: Section II gives a brief introduction to quantum statistics. Section III describes some existing models of networks as non-interacting particle systems. Section IV introduces the proposed model of network dynamics, based on Fermi-Dirac statistics, and shows the results of the corresponding simulations. Conclusions (i.e., Section V) end the paper.

## II. QUANTUM STATISTICS

Statistical mechanics assumes a central role when dealing with systems composed by many particles, the underlying assumption being that particles are identical and indistinguishable. Moreover, their quantum energy levels are extremely closely spaced, with a cardinality much greater than the number of particles. Energy levels can be grouped in bundles with the approximation that levels in the same bundle have the same energy. Particles with a symmetric wave function, called bosons, obey Bose-Einstein statistics, whereas particles with an antisymmetric wave function, called fermions, obey Fermi-Dirac statistics –see Ref. [12]. Given a system with  $N$  particles of the same type, we can build an  $N$ -body wave function having in an  $\alpha$  state a number of particles  $n_\alpha$ , called occupation number, such that:

$$n_\alpha = \begin{cases} 0, 1, \dots, \infty & \text{bosons} \\ 0, 1 & \text{fermions} \end{cases} \quad (2)$$

and  $\sum_{\alpha} n_{\alpha} = N$ . Considering a gas composed by  $N$  bosons, the number of microstates in such macrostate is computable by the following equation:

$$\Omega_b = \Pi_i \frac{(n_i + g_i)!}{n_i! g_i!} \quad (3)$$

with  $g_i$  representing the  $i$ -th bundle. The distribution of particles follows the Bose-Einstein statistics:

$$n_i^b = \frac{g_i}{e^{\frac{\epsilon_i - \mu}{k_b T}} - 1} \quad (4)$$

where  $\epsilon_i$  denotes the energy of the  $i$ -th bundle,  $\mu$  the chemical potential, and  $k_b$  the Boltzmann constant. In the event that a gas is composed by fermions, we must consider also the Pauli exclusion principle. Overall, the number of microstates is computed by the equation:

$$\Omega_f = \Pi_i \frac{g_i!}{n_i! (g_i - n_i)!} \quad (5)$$

The distribution of particles follows the Fermi-Dirac statistics:

$$n_i^f = \frac{g_i}{e^{\frac{\epsilon_i - \mu}{k_b T}} + 1} \quad (6)$$

Both these distributions approach the classical behaviour in proximity of the high-temperature limit, showing a quantum-classical transition. Such phenomenon occurs when particles occupy excited states sparsely. In particular, considering the thermal wavelength  $\lambda$  and the density  $\rho$ , the following conditions can be stated:

$$\begin{cases} \rho \lambda^3 \gg 1 & \text{classical regime} \\ \rho \lambda^3 \approx 1 & \text{onset of quantum effects} \end{cases} \quad (7)$$

The classical regime is described by the Maxwell-Boltzmann distribution. In particular, with  $Z = \sum_j g_j e^{-\frac{\epsilon_j}{k_b T}}$  partition function, we write:

$$n_i^{mb} = \frac{N}{Z} g_i e^{-\frac{\epsilon_i}{k_b T}} \quad (8)$$

### III. NETWORKS AS PARTICLE SYSTEMS

Let us give a synthetic overview of some existing models of network dynamics based on quantum statistics:

### A. Bosonic Networks

In this model, Bianconi and Barabasi Ref. [6] compared network evolution to phase transition of bosonic gases. Two main structures, i.e., *fit-get-rich* and *winner-takes-all*, are identified as two different phases at low temperatures. In this model, each node is interpreted as an energy level and each link as a pair of particles. Starting from a fitness parameter  $\eta$ , energy is computed according to the following equation:

$$\epsilon = -\frac{1}{\beta} \log \eta \quad (9)$$

with  $\beta = \frac{1}{T}$ . Here, the fitness parameter  $\eta$  describes the ability of each node to compete for new links. In particular, for the  $i$ -th node, the probability of connection with new nodes is proportional to:

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j} \quad (10)$$

with  $k_i$  degree of the  $i$ -th node. Notably, new nodes tend to link with pre-existing nodes having high values of  $(\eta, k)$ . The generation of a scale-free network in the *fit-get-rich* phase is characterized by Equation (1) and entails the presence of few nodes with a high degree connected to many others with low degree. In a bosonic gas, when the temperature decreases, particles aim to occupy lower energy levels until  $T \sim 0$ , then Bose-Einstein condensation takes place. In this model, as  $T$  decreases, many particles move to lower levels while keeping the corresponding particles at upper levels. In doing so, links concentrate on few nodes, until they condensate in the *winner-takes-all* phase, characterized by the fact that only one node predominates. In Ref. [7], Bianconi discussed the differences between bosonic and fermionic networks, showing that the former are scale-free, whereas the latter can be represented by Cayley trees.

### B. Fermi-Dirac Statistics of Complex Networks

In this work Ref. [9], the authors proposed an inverse approach to network evolution, starting from a random network with an average degree equal to  $2m$ . Using an *illness model*, at each step, one randomly-selected node becomes “ill”, then its illness causes a loss of links. An illness parameter  $I$  is defined and assigned randomly to each node. The probability for each node to lose a link depends on  $I$  and on the node degree. Hence, the

number of links decreases with time. Each node has an energy defined by:

$$\epsilon = -\frac{1}{\beta} \log I \quad (11)$$

with  $\beta = \frac{1}{k_b T}$ . A link between two nodes corresponds to a couple of non-interacting particles placed in two energy levels whose value is computed by Equation 11. When a new node joins the network, a new energy level is added, with  $m$  new particles, and other  $m$  links are deleted, due to the illness. Authors showed that the behavior of such system can be approximated by a Fermi distribution, so that the network can be seen as a Fermi gas.

#### IV. FERMIONIC NETWORKS

Let us now introduce a novel proposal for modelling network dynamics, based on Fermi-Dirac statistics. Given a network with  $n$  nodes and  $v$  edges, say  $E = (n, v)$ , let us represent each link as a particle and each node as a degenerate bundle of energy levels. Each energy level corresponds to a *potential*-link in the network, which is drawn only when at least one particle exists on it. The maximum number of links in a network is  $L = n(n-1)/2$ . Hence, in the limit case of a fully-connected network, the number of links is  $v = L$ . The number  $g_i$  of available states in the  $i$ -th bundle is much bigger than the number  $p_i$  of particles, i.e.,  $g_i \gg p_i$ . The  $i$ -th level has an energy  $\epsilon_i$ , which corresponds to that of the hosting bundle. This value can be assigned randomly or depends on a property of the system, e.g., a fitness parameter  $\eta$  or any other function deemed relevant (with the trivial constraint that it must be computable for each node of the network). In the proposed model, lower bundles contain more energy levels. In particular, the first bundle has  $n-1$  levels, the second has  $n-2$  levels, and so on and so forth. Note that the link  $l_{ij}$ , which connects nodes  $i$  and  $j$ , is represented only by a single energy level, i.e.,  $\epsilon_{ij}$ , which in turn is contained in the  $i$ -th bundle (under the assumption that the  $i$ -th bundle is deeper than the  $j$ -th). In doing so, the last node “ $y$ ” has no *potential*-links, although it can be linked in the event that a particle stays at the  $\epsilon_{xy}$  level, with  $x$  corresponding to one of the other nodes. Figure 1 shows an example aimed at graphically illustrating these concepts.

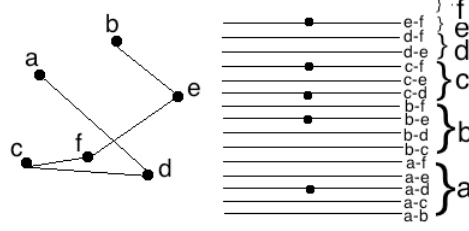


FIG. 1. On the left, a small network with 6 nodes and 5 links (each node is labeled by a letter). On the right, the corresponding fermionic model: a bundle  $g_i$  is assigned to each node, each *potential*-link has its proper energy level, and real links are represented by particles.

### A. Modelling Network Evolution

Let us consider a dynamical network  $E = (n(t), v(t))$ , with  $n(t)$  of nodes and  $v(t)$  edges, both time-varying (e.g., the world wide web or specific social networks). According to the *Fitness Model* Ref. [6], when a new node with fitness  $\eta$  joins the network, pre-existing nodes compete to connect with it.

For every node, a bundle is defined –whose energy is computed with Equation (9). The relative position of each bundle depends on the value of its energy, so that deeper bundles incorporate more available states. For each bundle, the number of available states increases if a new node with higher energy is added. Hence, nodes represented as bundles with low energy have more probability to gain new links. With  $t$  generic point in time, let the connectivity of the  $i$ -th node added at time  $t_i$  be  $k_i(\epsilon_i, t, t_i)$ . Every node generates  $m$  links when added to the network, hence  $m$  new particles are assigned to lower bundles. We also assume that, as time passes and other nodes join the network, the connectivity of the  $i$ -th node increases, in agreement with the scale-free model Ref. [13], according to the following power law:

$$k_i(\epsilon_i, t, t_i) = m \left( \frac{t}{t_i} \right)^{f(\epsilon_i)} \quad (12)$$

where  $f(\epsilon_i)$  is an energy dependent dynamic exponent, defined as:

$$f(\epsilon) = e^{-\beta\epsilon} \quad (13)$$

Considering the ability of particles to jump between energy levels, at high temperatures, the particle distribution follows the classical Maxwell-Boltzmann and particles are distributed among the available states –see Equation (8). As temperature decreases, many particles

move to lower energy levels so that when new particles join the system, most likely they will be located at low energy levels too –see Figure 2.

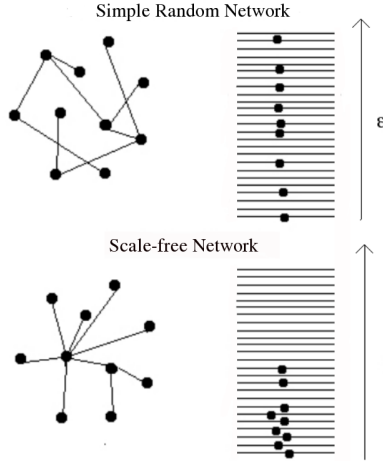


FIG. 2. On the left, from top to bottom, the evolution of a network with 10 nodes and 9 links from a simple random network to a scale-free network. On the right, their corresponding fermionic models, which result from a cooling process that pushes particles to low energy levels, contained in two bundles.

During this process, few nodes gain new links. Hence, when new nodes join the network, they have a higher probability to be selected, as their  $k_i$  is increased. Given the number of particles, it is possible to compute the Fermi Energy  $E_f$  of the system as the energy of the bundle containing the last particle at  $T \rightarrow 0$ . Hence, as temperature decreases, and assuming that the number of particles approximates the number of bundles, the *winner-takes-all* phase takes place. On the other hand, in the event that the number of particles grows, a *fit-get-rich* phase emerges (see also Ref. [6]). We mapped the temperature to the scale parameter of the gamma distribution used to perform a weighted random selection of nodes. This selection is executed when a new node joins the network and generates  $m$  new links with pre-existing nodes. According to the fermionic model previously described, it is a weighted random selection, as nodes represented as bundles with more available states have more chances to win. Finally, as shown in the following, it is possible to draw a rough separation in a  $T - \rho$  plane between the simple random and the scale-free region.



## B. Simulations of Fermionic Networks

The proposed fermionic model has been tested with many simulations. In particular, we generated networks of different size, varying the value of  $m$  and the system temperature  $T$ . Figure 3 shows three networks, with  $n = 50$  and  $m = 2$ , generated at different temperatures. At  $T = 0.5$  and  $T = 3$  the networks are scale-free, whereas at  $T = 10$  a simple random network is generated. Their degree distributions are represented in the  $P(k)$  diagram. For

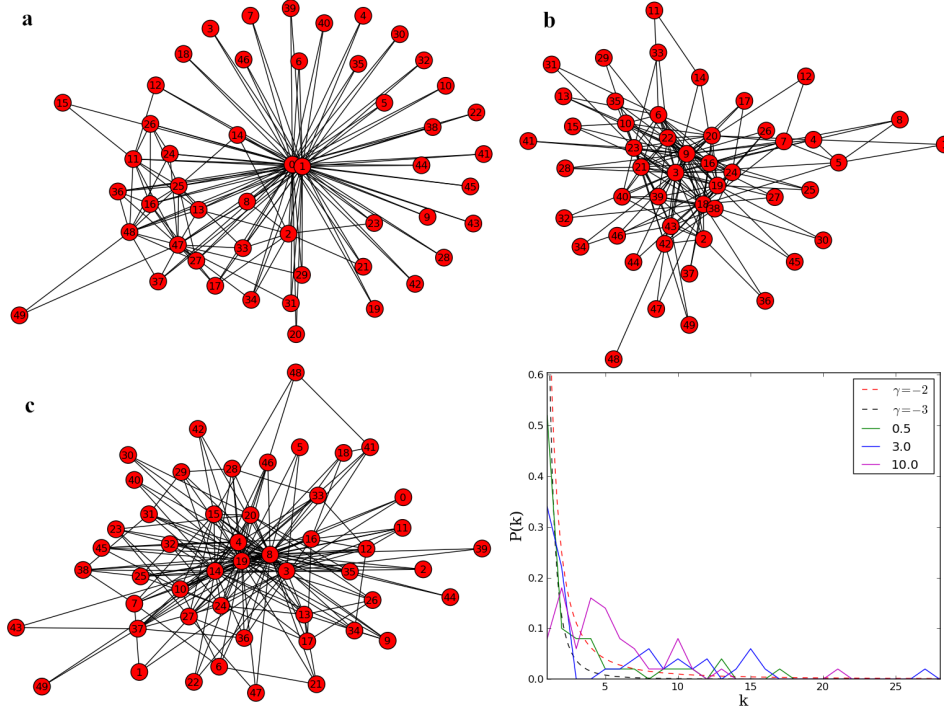


FIG. 3. The figure shows three fermionic networks, generated at different temperatures, as well as their degree distributions  $P(k)$ , with  $n = 50$  and  $m = 2$ . **a)** A fermionic network generated at  $T = 0.5$  (coloured in green). As shown in the diagram, its  $P(k)$  follows a power-law approximated by  $\gamma = 3$ . **b)** A fermionic network generated at  $T = 3$  (coloured in blue). As shown in the diagram, its  $P(k)$  follows a power-law approximated by  $\gamma \in (2, 3)$ . **c)** A fermionic network generated at  $T = 10$  (coloured in magenta). It is not scale-free, as its  $P(k)$  does not follow a power-law. For the sake of reference, two referential curves for scale-free networks are also shown in the  $P(k)$  diagram:  $\gamma = 2$  and  $\gamma = 3$ , see Equation (1), are represented by a black and red dotted line, respectively.

each simulation an approximated temperature of transition has been computed, from a scale-free structure to a simple random network. Figure 4 shows some results achieved using a

number of nodes from 250 to 2000, with different values of  $m$ . The temperature of transition,

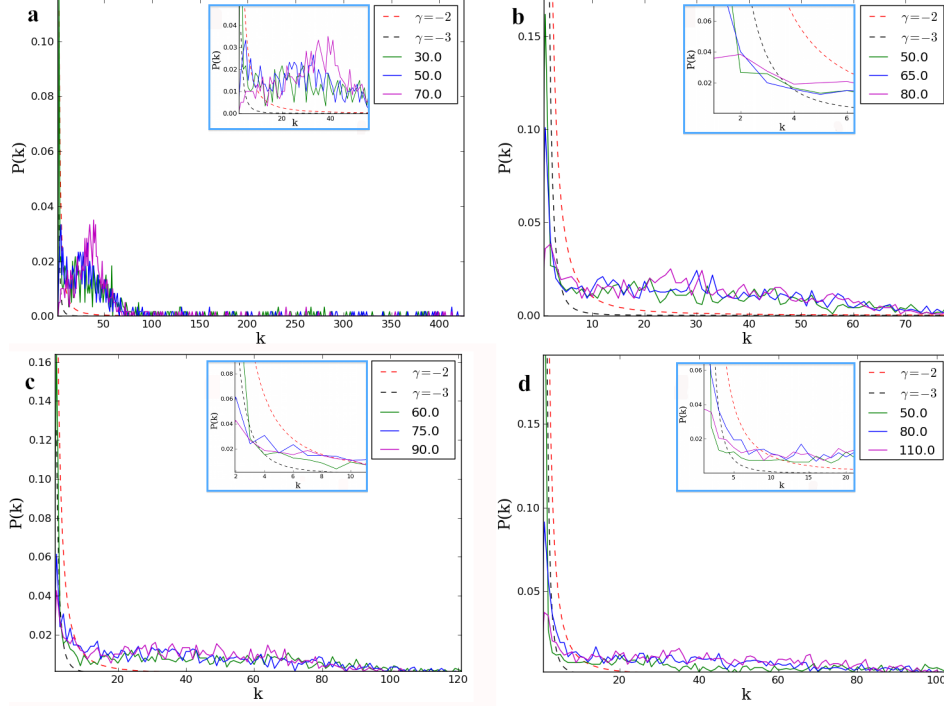


FIG. 4. Results of the degree distribution  $P(k)$  for some fermionic networks generated during the simulation. **a)** Fermionic network with  $n = 600$  and  $m = 4$ . At temperatures  $T > 40$  this network loses its scale-free structure. **b)** Fermionic network with  $n = 1200$  and  $m = 2$ . At temperatures  $T > 70$  this network loses its scale-free structure. **c)** Fermionic network with  $n = 1500$  and  $m = 3$ . At temperatures  $T > 75$  this network loses its scale-free structure. **d)** Fermionic network with  $n = 2000$  and  $m = 2$ . At temperatures  $T > 110$  this network loses its scale-free structure. Two referential curves for scale-free networks:  $\gamma = 2$  and  $\gamma = 3$ , see Equation (1), are represented by black and red dotted lines, respectively.

from a scale-free to a simple random structure, increases with the number of nodes. It is worth noting that these temperatures can be interpolated, in the  $T - \rho$  plane, by the curve  $\rho\lambda^3$ , with  $\rho$  number of particles (i.e., links of the network) and with  $\lambda = \alpha T^{-\frac{1}{2}}$  wavelength. The value  $\alpha = 0.2085$  best interpolates the temperatures – see Figure 5. This curve, for values less than the scale factor  $\alpha$ , mimics the curve that draws a rough separation between classical and quantum regime for quantum gases –see also Equation (7).

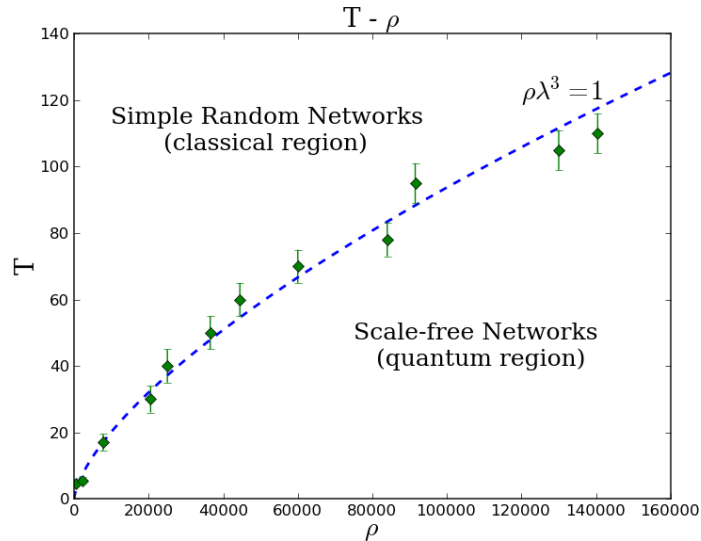


FIG. 5. Simple Random Networks and Scale-free Networks regions in the  $T - \rho$  plane. The dotted line is  $\rho\lambda^3$ , with  $\rho$  density and  $\lambda$  thermal wavelength. This line represents a rough separation between the two regions. The former corresponds to the classical region and the latter to the quantum region for quantum gases. Green  $\diamond$  represent the transition values  $(T, \rho)$  (with an error of  $\pm 5$ ) of the simulated fermionic networks from scale-free to simple random structures.

### C. Discussion

Fermionic networks show that the emergence of a scale-free structure can be represented as a quantum-classical transition for quantum gases. In particular, a scale-free network correspond to a fermionic gas approximated by the quantum regime at low temperatures. On the other hand, a simple random network corresponds to the same gas in classical regime at high temperatures. Similar considerations about the connection between classical random and scale-free networks have been proposed in Ref. [8]. The authors show that, in the cold regime, their network is scale-free, but as the temperature increases, the network loses its metric structure and its hierarchical heterogeneous organization –becoming a classical random network. Considering that many real complex networks are scale-free and others have not this structure, see Ref. [14], we deem that the proposed fermionic model can be considered a good candidate for representing their evolution, at low and high temperatures, respectively.

## V. CONCLUSIONS

In this work, we represent complex networks as quantum gases, thus defining a fermionic network model. Using this model, we show that network evolution is a temperature-dependent process. We found that it is possible to draw a separation between simple random region and scale-free region, in the  $T - \rho$  plane, with a curve similar to that used to separate classical and quantum regions for quantum gases. The transition from simple random to scale-free networks mimics a cooling process of a physical system achieving equilibrium, despite the non-equilibrium nature of network evolution. During this process, the transition from classical to quantum regime (or vice-versa for heating processes) takes place. Finally, we believe that many real complex networks can be mapped to quantum systems, at low or high temperatures depending on their structure. As for future work, we are planning to develop a model in which the temperatures can vary during network evolution.

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